Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

3 - 12 Residues
Find all the singularities in the finite plane and the corresponding residues.

3. $\frac{\text{Sin}[2\,z]}{z^6}$

```
Clear["Global`*"]
```

$\text{f1}[z\_] = \frac{\text{Sin}[2\,z]}{z^6}$

$\frac{\text{Sin}[2\,z]}{z^6}$

```
Residue[f1[z], 0] (* singularity at 0 *)
```

$\text{Residue}\left[\frac{\text{Sin}[2\,z]}{z^6}, \{z, 0\}\right]$

$\frac{4}{15}$

5. $\frac{8}{1+z^2}$

```
Clear["Global`*"]
```

$\text{f2}[z\_] = \frac{8}{1+z^2}$

$\frac{8}{1+z^2}$

The singularity is at $\pm i$.

I tried a few ways to get Mathematica to calculate both signs in a single step, but was unable to.

$-4\,i$

```
Residue[f2[z], {z, -i}]
```

4 i

7. Cot[π z]

```
Clear["Global`*"]

f3[z_] = Cot[π z]
Cot[π z]

exec[N_] = TableForm[Table[{n, f3[n]}, {n, -N, N}]]
```

Table::iterb : Iterator{n, −N, N} doesnothaveappropriatebounds ≫

```
Table[{n, f3[n]}, {n, -N, N}]
```

The problem function has singularities at multiples of z=1

```
exec[5]
-5    ComplexInfinity
-4    ComplexInfinity
-3    ComplexInfinity
-2    ComplexInfinity
-1    ComplexInfinity
0     ComplexInfinity
1     ComplexInfinity
2     ComplexInfinity
3     ComplexInfinity
4     ComplexInfinity
5     ComplexInfinity

Residue[f3[z], {z, 1}]
```

$\frac{1}{\pi}$

9. $\frac{1}{1 - e^z}$

```
Clear["Global`*"]
```

f4[z_] = $\frac{1}{1 - e^z}$

$\frac{1}{1 - e^z}$

The problem function has a singularity as described below.

```
Solve[1 - ⅇ^z == 0, z]
```

```
{{z → ConditionalExpression[2 ⅈ π C[1], C[1] ∈ Integers]}}
```

```
Residue[f4[z], {z, 2 π ⅈ}]
```

```
-1
```

11. $\frac{e^z}{(z - \pi i)^3}$

```
Clear["Global`*"]
```

$$f5[z\_] = \frac{e^z}{(z - \pi i)^3}$$

$$\frac{e^z}{(-i\pi + z)^3}$$

The problem function has a singularity at the multiple root shown below,

$$\text{Solve}\left[(-i\pi + z)^3 == 0, z\right]$$

```
{{z → ⅈ π}, {z → ⅈ π}, {z → ⅈ π}}
```

$$\text{Residue}\left[\frac{e^z}{(z - \pi i)^3}, \{z, \pi i\}\right]$$

$$-\frac{1}{2}$$

14 - 25 Residue integration
Evaluate (counterclockwise).

15. $\oint_C \text{Tan}[2\pi z]\, dz$, C : $|z - 0.2| = 0.2$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[Cos[2 π z] == 0, z]
```

$$\left\{\left\{z \to \text{ConditionalExpression}\left[\frac{-\frac{\pi}{2} + 2\pi C[1]}{2\pi}, C[1] \in \text{Integers}\right]\right\},\right.$$
$$\left.\left\{z \to \text{ConditionalExpression}\left[\frac{\frac{\pi}{2} + 2\pi C[1]}{2\pi}, C[1] \in \text{Integers}\right]\right\}\right\}$$

I can put together a plot of the path of the integral.

```
p4 = ParametricPlot[{0.2 Cos[t] + 0.2, 0.2 Sin[t]}, {t, -π, π}, ImageSize →
    200, Epilog -> {PointSize[0.013], Point[{{0.2, 0}, {1, 3}}]},
   AxesLabel → {"Re", "Im"}, PlotRange → {-1, 1}];
```

And generate a plot of the singularities.

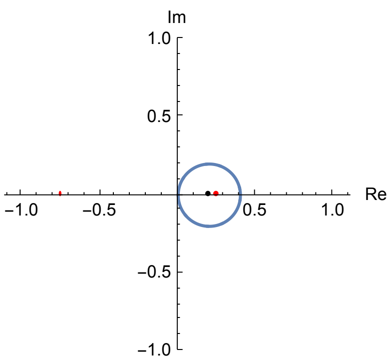
```
p1p = ListPlot[
   Table[{Re[(π/2 + 2 π n)/(2 π)], Im[(π/2 + 2 π n)/(2 π)]}, {n, -5, 5}], PlotStyle → {Red}];
```

Since in this case the singularities have only real components, the two list plots have the same output, and it is not necessary to deal with them both.

```
p2p = ListPlot[Table[{Re[(π/2 + 2 π n)/(2 π)], Im[(-π/2 + 2 π n)/(2 π)]}, {n, -5, 5}]];
```

There is only one singularity inside the integral's path, the one at z=1/4+0i. I can show both plots together.

```
Show[p4, p1p]
```




```
Residue[Tan[2 π z], {z, 1 / 4}]
```

$$-\frac{1}{2\pi}$$

According to numbered line (6) on p. 723,

$$\oint_C f[z]\, dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{Residue}}[f[z]]$$ where j to k covers the number of relevant singularities

So in this case it is just a matter of

$$2\pi i \left(-\frac{1}{2\pi}\right)$$

$$-i$$

17. $\oint_c \frac{e^z}{\text{Cos}[z]} dz$, C : $\left| z - \frac{\pi i}{2} \right| = 4.5$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[Cos[z] == 0, z]
```

```
{{z → ConditionalExpression[-π/2 + 2 π C[1], C[1] ∈ Integers]},
 {z → ConditionalExpression[π/2 + 2 π C[1], C[1] ∈ Integers]}}
```

I can put together a plot of the path of the integral.

```
p5 = ParametricPlot[{4.5 Cos[t], 4.5 Sin[t] + π/2}, {t, -π, π},
   ImageSize → 200, Epilog -> {PointSize[0.013], Point[{{0, π/2}}]},
   AxesLabel → {"Re", "Im"}, PlotRange → {-10, 10},
   PlotStyle → {Thickness[0.004]}];
```

And generate a plot of the singularities.

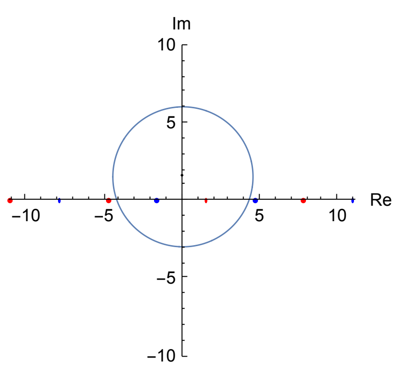
```
p1p = ListPlot[
   Table[{Re[π/2 + 2 n π], Im[π/2 + 2 n π]}, {n, -10, 10}], PlotStyle → {Red}];
```

This time, since the plotted points differ, it is necessary to look at both plots.

```
p2p = ListPlot[Table[{Re[-π/2 + 2 n π], Im[-π/2 + 2 n π]}, {n, -10, 10}],
   PlotStyle → {Blue}];
```

And show both plots together.

```
Show[p5, p1p, p2p]
```



The relevant singularities occur when n = 0, thus at $-\frac{\pi}{2} + 0\,i$ and $\frac{\pi}{2} + 0i$

```
Residue[e^z/Cos[z], {z, -π/2}]
```

$e^{-\pi/2}$

```
Residue[e^z/Cos[z], {z, π/2}]
```

$-e^{\pi/2}$

So this time the assemblage will be equal to

$2\,\pi\,i\,\left(e^{-\pi/2} + -e^{\pi/2}\right)$

$2\,i\,\left(e^{-\pi/2} - e^{\pi/2}\right)\,\pi$

```
FullSimplify[%]
```

$-4\,i\,\pi\,\text{Sinh}\left[\frac{\pi}{2}\right]$

19. $\oint_C \frac{\text{Sinh}[z]}{2\,z - i}\,dz$, C : $|\,z - 2\,i\,| = 2$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[2 z - i == 0, z]
```

$\left\{\left\{z \to \frac{i}{2}\right\}\right\}$

I can put together a plot of the path of the integral.

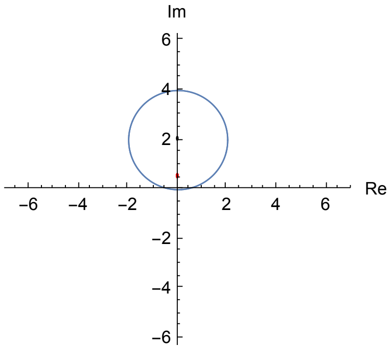
```
p7 = ParametricPlot[{2 Cos[t], 2 Sin[t] + 2}, {t, -2 π, 2 π},
   ImageSize → 200, Epilog -> {PointSize[0.013], Point[{{0, 2}}]},
   AxesLabel → {"Re", "Im"}, PlotRange → {-2 π, 2 π},
   PlotStyle → {Thickness[0.004]}];
```

And generate a plot of the singularities.

```
p1p = ListPlot[Table[{Re[i/2], Im[i/2]}, {n, -10, 10}], PlotStyle → {Red}];
```

And show both plots together.

```
Show[p7, p1p]
```



The sole singularity is relevant.

$$\text{Residue}\left[\frac{\text{Sinh}[z]}{2\,z - i}, \left\{z, \frac{i}{2}\right\}\right]$$

$$\frac{1}{2}\, i\, \text{Sin}\left[\frac{1}{2}\right]$$

Oddly, the residue calculated by Mathematica does not agree with the text answer. The text answer looks the same, except $\text{Sin}[\frac{1}{2}]$ is shown as $\text{Sinh}[\frac{1}{2}]$. Comparing the two for a possible typo,

```
N[Sin[1 / 2]]
```

0.479426

```
N[Sinh[1 / 2]]
```

0.521095

Hmm. Anyway, this time the developed answer will be equal to

$$2\,\pi\, i\left(\frac{1}{2}\, i\, \text{Sin}\left[\frac{1}{2}\right]\right)$$

$$-\pi\, \text{Sin}\left[\frac{1}{2}\right]$$

And in this answer, agreement is found with that of the text.

21. $\oint_C \frac{\text{Cos}[\pi\, z]}{z^5}\, dz$, C : $|z| = \frac{1}{2}$

```
Clear["Global`*"]
```

First I will investigate the singularities.

$$\text{Solve}\left[z^5 == 0, z\right]$$

{{z → 0}, {z → 0}, {z → 0}, {z → 0}, {z → 0}}

I can put together a plot of the path of the integral.

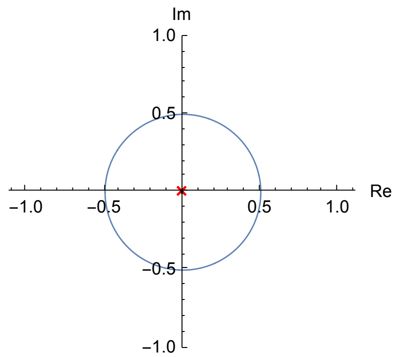
```
p8 = ParametricPlot[{1/2 Cos[t], 1/2 Sin[t]}, {t, -π, π},
   ImageSize → 200, Epilog -> {PointSize[0.013], Point[{{0, 0}}]},
   AxesLabel → {"Re", "Im"}, PlotRange → {-1, 1},
   PlotStyle → {Thickness[0.004]}];
```

And generate a plot of the singularities.

```
p1p = ListPlot[Table[{Re[0], Im[0]}, {n, -2, 2}],
   PlotStyle → {Red}, PlotMarkers → {"×", 12}];
```

And show both plots together.

```
Show[p8, p1p]
```



The sole singularity is relevant.

```
Residue[Cos[π z]/z^5, {z, 0}]
```

$$\frac{\pi^4}{24}$$

Another case where the residue as calculated by Mathematica does not match exactly with the text answer. In this case the text answer shows a z factor in the denominator. Ignoring this snag, the final assemblage will be equal to

$$2\,\pi\,i\left(\frac{\pi^4}{24}\right)$$

$$\frac{i\,\pi^5}{12}$$

The above answer matches that of the text.

23. $\oint_C \frac{30\,z^2 - 23\,z + 5}{(2\,z-1)^2\,(3\,z-1)}\,dz$, C : the unit circle

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[(2 z - 1)^2 (3 z - 1) == 0, z]
```

$\{\{z \to \frac{1}{3}\}, \{z \to \frac{1}{2}\}, \{z \to \frac{1}{2}\}\}$

I can put together a plot of the path of the integral.

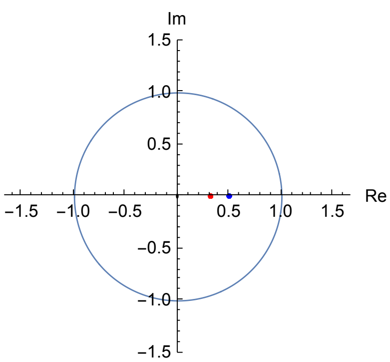
```
p9 = ParametricPlot[{1 Cos[t], 1 Sin[t]}, {t, -π, π},
   ImageSize → 200, Epilog -> {PointSize[0.013], Point[{{0, 0}}]},
   AxesLabel → {"Re", "Im"}, PlotRange → {-1.5, 1.5},
   PlotStyle → {Thickness[0.004]}];
```

And generate a plot of the singularities.

```
p1p = ListPlot[Table[{Re[1/3], Im[1/3]}, {n, -10, 10}], PlotStyle → {Red}];

p2p = ListPlot[Table[{Re[1/2], Im[1/2]}, {n, -10, 10}], PlotStyle → {Blue}];
```

And show three plots together.

```
Show[p9, p1p, p2p]
```



Evidently two singularities are relevant.

$$\text{Residue}\left[\frac{30 z^2 - 23 z + 5}{(2 z - 1)^2 (3 z - 1)}, \{z, \frac{1}{3}\}\right]$$

2

$$\text{Residue}\left[\frac{30 z^2 - 23 z + 5}{(2 z - 1)^2 (3 z - 1)}, \{z, \frac{1}{2}\}\right]$$

$\frac{1}{2}$

The consolidated answer is

$$2\pi i\left(2+\frac{1}{2}\right)$$

$$5 i \pi$$

Green cells match text answers.

$$25.\ \oint_C \frac{z\,\mathrm{Cosh}[\pi z]}{z^4+13z^2+36}\,dz\ ,\ C:\ |z| = \pi$$

```
Clear["Global`*"]
```

First I will investigate the singularities.

```
Solve[z^4 + 13 z^2 + 36 == 0, z]
{{z → -2 i}, {z → 2 i}, {z → -3 i}, {z → 3 i}}
```

I can put together a plot of the path of the integral.

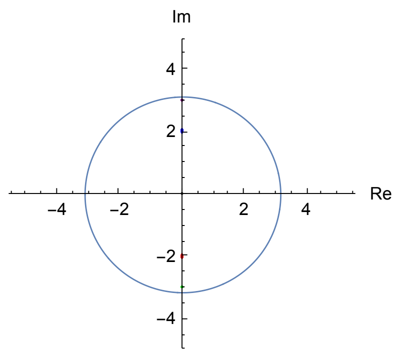
```
p10 = ParametricPlot[{π Cos[t], π Sin[t]}, {t, -π, π},
   ImageSize → 200, Epilog -> {PointSize[0.013], Point[{{0, 0}}]},
   AxesLabel → {"Re", "Im"}, PlotRange → {-5, 5},
   PlotStyle → {Thickness[0.004]}];
```

And generate plots of the singularities.

```
p1p =
  ListPlot[Table[{Re[-2 i], Im[-2 i]}, {n, -10, 10}], PlotStyle → {Red}];

p2p = ListPlot[Table[{Re[2 i], Im[2 i]}, {n, -10, 10}], PlotStyle → {Blue}];

p3p = ListPlot[
   Table[{Re[-3 i], Im[-3 i]}, {n, -10, 10}], PlotStyle → {Green}];

p4p =
  ListPlot[Table[{Re[3 i], Im[3 i]}, {n, -10, 10}], PlotStyle → {Purple}];
```

And show five plots together.

```
Show[p10, p1p, p2p, p3p, p4p]
```



The combined plot shows four relevant singularities.

$$\text{Residue}\left[\frac{z\,\text{Cosh}[\pi z]}{z^4 + 13 z^2 + 36}, \{z, -2 i\}\right]$$

$$\frac{1}{10}$$

$$\text{Residue}\left[\frac{z\,\text{Cosh}[\pi z]}{z^4 + 13 z^2 + 36}, \{z, 2 i\}\right]$$

$$\frac{1}{10}$$

$$\text{Residue}\left[\frac{z\,\text{Cosh}[\pi z]}{z^4 + 13 z^2 + 36}, \{z, -3 i\}\right]$$

$$\frac{1}{10}$$

$$\text{Residue}\left[\frac{z\,\text{Cosh}[\pi z]}{z^4 + 13 z^2 + 36}, \{z, 3 i\}\right]$$

$$\frac{1}{10}$$

So the consolidated answer will be

$$2 \pi i \left(\frac{4}{10}\right)$$

$$\frac{4 i \pi}{5}$$